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# Pretwisted Nonuniform Beams with Time-Dependent Elastic Boundary Conditions

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**The governing differential equations with the nonhomogeneous time-dependent elastic boundary conditions for the coupled bending–bending vibration of a pretwisted nonuniform beam are derived by Hamilton's principle. By taking a general change of dependent variable with shifting functions, the original system is transformed to be a system composed of two nonhomogeneous governing differential equations and eight homogeneous boundary conditions. Consequently, the method of separation of variables can be used to solve the transformed problem. The physical meanings of these shifting functions are explored. The orthogonality condition for the eigenfunctions of a pretwisted nonuniform beam with elastic boundary conditions is also derived. The stiffness matrix for a nonuniform beam with arbitrary pretwist is derived. As the coefficients of the matrix can be integrated analytically, the exact stiffness matrix is, therefore, obtained. The relation between the shifting functions and the stiffness matrix is derived. The influence of the pretwist angle on the dynamic response of the beam is studied. The vibration control of a pretwist beam with boundary inputs is investigated.**

## Nomenclature

| | |
|---|---|
| $A$ | = cross-sectional area of the beam |
| $B_{ij}$ | = dimensionless bending rigidity, $E(x)I_{ij}(x)/[E(0)I_{yy}(0)]$, $i, j = x, y$ |
| $E$ | = Young's modulus of beam material |
| $F_1, F_2, F_3, F_4$, $F_5, F_6, F_7, F_8$, $F_1^*, F_2^*, F_3^*, F_4^*$, $F_5^*, F_6^*, F_7^*, F_8^*$ | = dimensionless slopes, displacements, external moments, and shear excitations at the left and right of the beam in the $y$ and $z$ directions, respectively, $f_1$, $f_2/L$, $f_3$, $f_4/L$, $f_5$, $f_6/L$, $f_7$, $f_8/L$, $f_1^*L/[E(0)I_{yy}(0)]$, $f_2^*L^2/[E(0)I_{yy}(0)]$, $f_3^*L/[E(0)I_{yy}(0)]$, $f_4^*L^2/[E(0)I_{yy}(0)]$, $f_5^*L/[E(0)I_{yy}(0)]$, $f_6^*L^2/[E(0)I_{yy}(0)]$, $f_7^*L/[E(0)I_{yy}(0)]$, $f_8^*L^2/[E(0)I_{yy}(0)]$ |
| $f_1, f_2, f_3, f_4$, $f_5, f_6, f_7, f_8$, $f_1^*, f_2^*, f_3^*, f_4^*$, $f_5^*, f_6^*, f_7^*, f_8^*$ | = slopes, displacements, external moments, and shear excitations at the left and right of the beam in the $y$ and $z$ directions, respectively |
| $I$ | = area moment inertia of the beam |
| $K_{yTL}, K_{y\theta L}$, $K_{yTR}, K_{y\theta R}$, $K_{zTL}, K_{z\theta L}$, $K_{zTR}, K_{z\theta R}$ | = translational and rotational spring constants at the left and the right end of the beam in the $y$ and $z$ directions, respectively |
| $L$ | = length of the beam |
| $M$ | = dimensionless mass, $\rho(x)A(x)/[\rho(0)A(0)]$ |
| $P, Q$ | = dimensionless external transverse loads in the $y$ and $z$ directions, respectively, $p(x,t)L^3/[E(0)I_{yy}(0)]$, $q(x,t)L^3/[E(0)I_{yy}(0)]$ |
| $p, q$ | = external transverse loads in the $y$ and $z$ directions, respectively |
| $T$ | = kinetic energy |
| $t$ | = time variable |
| $u, v, w$ | = beam lateral displacements in the $x$, $y$, and $z$ directions, respectively |
| $V, W$ | = dimensionless lateral displacements in the $y$ and $z$ directions, respectively, $v/L$, $w/L$ |
| $X, Y, Z$ | = principal frame coordinates |
| $x, y, z$ | = fixed frame coordinates |
| $\beta_1, \beta_2, \beta_3, \beta_4$, $\beta_5, \beta_6, \beta_7, \beta_8$ | = dimensionless rotational and translational spring constants at the left and right of the beam in the $y$ and $z$ directions, respectively, $K_{z\theta L}L/[E(0)I_{yy}(0)]$, $K_{zTL}L^3/[E(0)I_{yy}(0)]$, $K_{y\theta L}L/[E(0)I_{yy}(0)]$, $K_{yTL}L^3/[E(0)I_{yy}(0)]$, $K_{z\theta R}L/[E(0)I_{yy}(0)]$, $K_{zTR}L^3/[E(0)I_{yy}(0)]$, $K_{y\theta R}L/[E(0)I_{yy}(0)]$, $K_{yTR}L^3/[E(0)I_{yy}(0)]$ |
| $\theta$ | = angle between principal and fixed frames |
| $\Lambda^2$ | = dimensionless natural frequency, $m(0)\Omega^2L^4/[E(0)I_{yy}(0)]$ |
| $\xi$ | = dimensionless distance to the left end of the beam, $x/L$ |
| $\Pi$ | = total potential energy |
| $\rho$ | = mass density per unit volume |
| $\sigma_{xx}, \varepsilon_{xx}$ | = normal stress and strain in the $x$ direction, respectively |
| $\tau$ | = dimensionless time, $(t/L^2)\sqrt{[E(0)I_{yy}(0)/\rho(0)A(0)]}$ |
| $\Phi$ | = tip pretwist angle of the beam, $\theta(L)$ |
| $\Omega$ | = natural frequency |
| $\varpi$ | = dimensionless excitation frequency |

## Introduction

THE dynamic analysis of the pretwisted beams is important in the design of a number of engineering components, e.g., turbine blades, helicopter rotor blades, and gear teeth. An interesting review of the subject can be found in the paper by Rosen.[1] The forced vibration problem of a pretwisted nonuniform beam with general elastic time-dependent boundary conditions is common in engineering applications. Thus, it is necessary to develop an accurate and simple method to solve this complicated problem and to find its performance.

The vibrations of unpretwisted uniform Bernoulli–Euler beams with classical time-dependent boundary conditions can be solved by using the method of Laplace transform[2] and the method of Mindlin–Goodman[3] (also see Ref. 4). In the Mindlin–Goodman method, a change of dependent variable together with four shifting polynomial functions of fifth order is introduced. In general, by properly selecting these shifting polynomial functions, the original system will be transformed to a system composed of a nonhomogeneous governing differential equation with four homogeneous boundary conditions. Consequently, the method of separation of variables can be used to solve the problem. The dynamic analysis of a nonuniform Bernoulli–Euler beam with general time-dependent boundary

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conditions was given by Lee and Lin.[5] They generalized the method of Mindlin–Goodman and introduced four shifting functions with the physical meaning instead of those functions with no physical meaning given by Mindlin and Goodman.[3] The vibrations of unpretwisted uniform Timoshenko beams with classical time-dependent boundary conditions were studied by Herrmann[6] and Berry and Nagdhi[7] by using the method of Mindlin–Goodman. Lee and Lin[8] extended their previous study[5] and further generalized the method of Mindlin–Goodman to develop a solution procedure for studing the vibrations of an unpretwisted nonuniform Timoshenko beam with general time-dependent boundary conditions.

Approximation methods are very useful tools to investigate the free vibrations of pretwisted beams, where exact solutions are difficult to obtain even for the simplest cases. These methods are the finite element method,[9,10] the Rayleigh–Ritz method,[11] the Reissner method,[12] the Galerkin method,[13] and the transfer matrix method.[14–16] Rosard and Lester[14] assumed that the displacements at each element were linear. Rao and Carnegie[15] used an iteration procedure to determine the displacements at each element. Lin[16] derived the exact field transfer matrix of a nonuniform pretwisted beam with arbitrary pretwist without an iteration procedure and studied the performance of a beam with elastic boundary conditions. No research has been devoted to the forced vibration of the pretwisted beam with time-dependent elastic boundary conditions.

In this paper, the previous studies by Lee and Lin[8] and Lin[16] are extended. The governing differential equations with time-dependent elastic boundary conditions for the coupled bending–bending vibration of a pretwisted nonuniform beam are derived by Hamilton's principle. A solution procedure for studying the dynamic behavior of the system is developed by using the method of Mindlin–Goodman, and the eigensolutions of the system are obtained by using the modified transfer method given by Lin.[16] A general change of dependent variable with shifting functions is introduced, and the physical meanings of these shifting functions are further explored. The orthogonality condition for the eigenfunctions of a nonuniform pretwisted beam with elastic boundary conditions is also derived. The stiffness matrix for a nonuniform beam with arbitrary pretwist is derived. The relation between the shifting functions and the stiffness matrix is derived. The influence of the pretwist angle on the dynamic response of the beam is studied. The vibration control of a pretwist beam with boundary inputs is investigated.

## Governing Equations and Boundary Conditions

Consider the forced vibration problem of a pretwisted nonuniform beam with time-dependent elastic boundary conditions as shown in Fig. 1. Neither shear deformation nor rotatory inertia is considered. The displacement fields of the beam are

$$u = -z\frac{\partial w}{\partial x} - y\frac{\partial v}{\partial x}, \qquad v = v(x,t), \qquad w = w(x,t) \tag{1}$$

The total potential energy $\Pi$ and the kinetic energy $T$ of the beam are

$$\begin{aligned}
\Pi = {} & \frac{1}{2}\int_0^L \int_A \sigma_{xx}\varepsilon_{xx}\,\mathrm{d}A\,\mathrm{d}x + \frac{1}{2}K_{z\theta L}\left[\frac{\partial w(0,t)}{\partial x} - f_1(t)\right]^2 \\
& + \frac{1}{2}K_{zTL}[w(0,t) - f_2(t)]^2 + \frac{1}{2}K_{y\theta L}\left[\frac{\partial v(0,t)}{\partial x} - f_3(t)\right]^2 \\
& + \frac{1}{2}K_{yTL}[v(0,t) - f_4(t)]^2 + \frac{1}{2}K_{z\theta R}\left[\frac{\partial w(L,t)}{\partial x} - f_5(t)\right]^2 \\
& + \frac{1}{2}K_{zTR}[w(L,t) - f_6(t)]^2 + \frac{1}{2}K_{y\theta R}\left[\frac{\partial v(L,t)}{\partial x} - f_7(t)\right]^2 \\
& + \frac{1}{2}K_{yTR}[v(L,t) - f_8(t)]^2 \\
& - \int_0^L [p(x,t)w(x,t) + q(x,t)v(x,t)]\,\mathrm{d}x \\
& - f_1^*(t)\frac{\partial w(0,t)}{\partial x} - f_2^*(t)w(0,t) - f_3^*(t)\frac{\partial v(0,t)}{\partial x} \\
& - f_4^*(t)v(0,t) - f_5^*(t)\frac{\partial w(L,t)}{\partial x} - f_6^*(t)w(L,t) \\
& - f_7^*(t)\frac{\partial v(L,t)}{\partial x} - f_8^*(t)v(L,t)
\end{aligned} \tag{2}$$

$$T = \frac{1}{2}\int_0^L \int_A \left[\left(\frac{\partial w}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2\right]\rho\,\mathrm{d}A\,\mathrm{d}x$$

Application of Hamilton's principle yields two coupled governing differential equations and eight time-dependent elastic boundary conditions.

The two coupled dimensionless governing differential equations of the system are written as

$$\frac{\partial^2}{\partial\xi^2}\left(B_{yy}\frac{\partial^2 W}{\partial\xi^2}\right) + \frac{\partial^2}{\partial\xi^2}\left(B_{yz}\frac{\partial^2 V}{\partial\xi^2}\right) + M\frac{\partial^2 W}{\partial\tau^2} = P(\xi,\tau) \tag{3}$$

$$\frac{\partial^2}{\partial\xi^2}\left(B_{zz}\frac{\partial^2 V}{\partial\xi^2}\right) + \frac{\partial^2}{\partial\xi^2}\left(B_{yz}\frac{\partial^2 W}{\partial\xi^2}\right) + M\frac{\partial^2 V}{\partial\tau^2} = Q(\xi,\tau)$$

$$\xi \in (0,1) \tag{4}$$

and the associated dimensionless time-dependent elastic boundary conditions are given follows.

At $\xi = 0$,

$$\gamma_{11}\frac{\partial W}{\partial\xi} - \gamma_{12}\left(B_{yy}\frac{\partial^2 W}{\partial\xi^2} + B_{yz}\frac{\partial^2 V}{\partial\xi^2}\right) = \bar{F}_1(\tau) \tag{5}$$

$$\gamma_{21}W + \gamma_{22}\left[\frac{\partial}{\partial\xi}\left(B_{yy}\frac{\partial^2 W}{\partial\xi^2}\right) + \frac{\partial}{\partial\xi}\left(B_{yz}\frac{\partial^2 V}{\partial\xi^2}\right)\right] = \bar{F}_2(\tau) \tag{6}$$

$$\gamma_{31}\frac{\partial V}{\partial\xi} - \gamma_{32}\left(B_{zz}\frac{\partial^2 V}{\partial\xi^2} + B_{yz}\frac{\partial^2 W}{\partial\xi^2}\right) = \bar{F}_3(\tau) \tag{7}$$

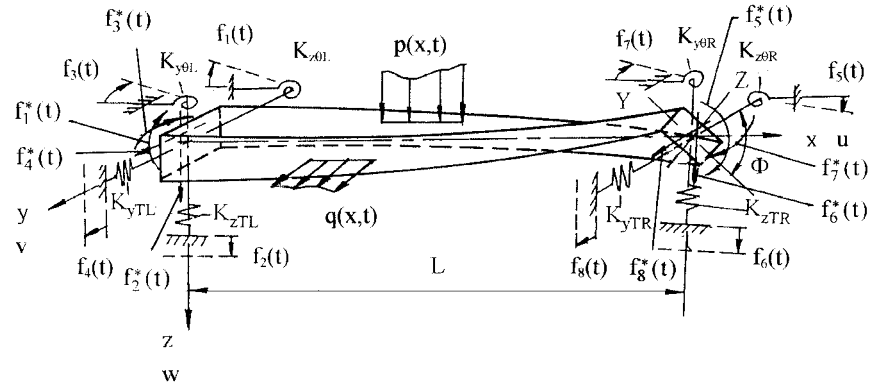


**Fig. 1 Geometry and coordinate system of a pretwisted nonuniform beam with time-dependent elastic boundary conditions.**

$$\gamma_{41}V+\gamma_{42}\left[\frac{\partial}{\partial\xi}\left(B_{zz}\frac{\partial^2 V}{\partial\xi^2}\right)+\frac{\partial}{\partial\xi}\left(B_{yz}\frac{\partial^2 W}{\partial\xi^2}\right)\right]=\bar{F}_4(\tau) \tag{8}$$

At $\xi=1$,

$$\gamma_{51}\frac{\partial W}{\partial\xi}+\gamma_{52}\left(B_{yy}\frac{\partial^2 W}{\partial\xi^2}+B_{yz}\frac{\partial^2 V}{\partial\xi^2}\right)=\bar{F}_5(\tau) \tag{9}$$

$$\gamma_{61}W-\gamma_{62}\left[\frac{\partial}{\partial\xi}\left(B_{yy}\frac{\partial^2 W}{\partial\xi^2}\right)+\frac{\partial}{\partial\xi}\left(B_{yz}\frac{\partial^2 V}{\partial\xi^2}\right)\right]=\bar{F}_6(\tau) \tag{10}$$

$$\gamma_{71}\frac{\partial V}{\partial\xi}+\gamma_{72}\left(B_{zz}\frac{\partial^2 V}{\partial\xi^2}+B_{yz}\frac{\partial^2 W}{\partial\xi^2}\right)=\bar{F}_7(\tau) \tag{11}$$

$$\gamma_{81}V-\gamma_{82}\left[\frac{\partial}{\partial\xi}\left(B_{zz}\frac{\partial^2 V}{\partial\xi^2}\right)+\frac{\partial}{\partial\xi}\left(B_{yz}\frac{\partial^2 W}{\partial\xi^2}\right)\right]=\bar{F}_8(\tau) \tag{12}$$

where

$$\bar{F}_i(\tau)=\gamma_{i1}F_i(\tau)+\gamma_{i2}F_i^*(\tau)$$
$$\gamma_{i1}=[\beta_i/(1+\beta_i)],\qquad \gamma_{i2}=[1/(1+\beta_i)] \tag{13}$$

When the dimensionless translational spring constant is infinity or zero, the time-dependent displacement or the time-dependent shear force is prescribed. If the dimensionless rotational spring constant is infinity or zero, the time-dependent slope or the time-dependent moment is prescribed.

The associated dimensionless initial conditions of the motion are

$$W(\xi,0)=W_0(\xi),\qquad V(\xi,0)=V_0(\xi)$$
$$\frac{\partial W(\xi,0)}{\partial\tau}=\dot{W}_0(\xi),\qquad \frac{\partial V(\xi,0)}{\partial\tau}=\dot{V}_0(\xi) \tag{14}$$

## Solution Method

### Change of Variable

To find the solution for these differential equations with variable coefficients and nonhomogeneous elastic boundary conditions, one generalizes the method given by Lee and Lin[5] by taking

$$W(\xi,\tau)=\bar{w}(\xi,\tau)+\sum_{i=1}^{8}\bar{F}_i(\tau)g_i(\xi)$$
$$V(\xi,\tau)=\bar{v}(\xi,\tau)+\sum_{i=1}^{8}\bar{F}_i(\tau)\bar{g}_i(\xi) \tag{15}$$

where the shifting functions $g_i(\xi)$ and $\bar{g}_i(\xi)$ are chosen to satisfy the following two differential equations:

$$\frac{d^2}{d\xi^2}\left(B_{yy}\frac{d^2 g_i}{d\xi^2}\right)+\frac{d^2}{d\xi^2}\left(B_{yz}\frac{d^2\bar{g}_i}{d\xi^2}\right)=0 \tag{16}$$

$$\frac{d^2}{d\xi^2}\left(B_{zz}\frac{d^2\bar{g}_i}{d\xi^2}\right)+\frac{d^2}{d\xi^2}\left(B_{yz}\frac{d^2 g_i}{d\xi^2}\right)=0,\qquad i=1,2,\ldots,8 \tag{17}$$

and the following boundary conditions.

At $\xi=0$,

$$\gamma_{11}\frac{dg_i}{d\xi}-\gamma_{12}\left(B_{yy}\frac{d^2 g_i}{d\xi^2}+B_{yz}\frac{d^2\bar{g}_i}{d\xi^2}\right)=\delta_{i1} \tag{18}$$

$$\gamma_{21}g_i+\gamma_{22}\left[\frac{d}{d\xi}\left(B_{yy}\frac{d^2 g_i}{d\xi^2}\right)+\frac{d}{d\xi}\left(B_{yz}\frac{d^2\bar{g}_i}{d\xi^2}\right)\right]=\delta_{i2} \tag{19}$$

$$\gamma_{31}\frac{d\bar{g}_i}{d\xi}-\gamma_{32}\left(B_{zz}\frac{d^2\bar{g}_i}{d\xi^2}+B_{yz}\frac{d^2 g_i}{d\xi^2}\right)=\delta_{i3} \tag{20}$$

$$\gamma_{41}\bar{g}_i+\gamma_{42}\left[\frac{d}{d\xi}\left(B_{zz}\frac{d^2\bar{g}_i}{d\xi^2}\right)+\frac{d}{d\xi}\left(B_{yz}\frac{d^2 g_i}{d\xi^2}\right)\right]=\delta_{i4} \tag{21}$$

At $\xi=1$,

$$\gamma_{51}\frac{dg_i}{d\xi}+\gamma_{52}\left(B_{yy}\frac{d^2 g_i}{d\xi^2}+B_{yz}\frac{d^2\bar{g}_i}{d\xi^2}\right)=\delta_{i5} \tag{22}$$

$$\gamma_{61}g_i-\gamma_{62}\left[\frac{d}{d\xi}\left(B_{yy}\frac{d^2 g_i}{d\xi^2}\right)+\frac{d}{d\xi}\left(B_{yz}\frac{d^2\bar{g}_i}{d\xi^2}\right)\right]=\delta_{i6} \tag{23}$$

$$\gamma_{71}\frac{d\bar{g}_i}{d\xi}+\gamma_{72}\left(B_{zz}\frac{d^2\bar{g}_i}{d\xi^2}+B_{yz}\frac{d^2 g_i}{d\xi^2}\right)=\delta_{i7} \tag{24}$$

$$\gamma_{81}\bar{g}_i-\gamma_{82}\left[\frac{d}{d\xi}\left(B_{zz}\frac{d^2\bar{g}_i}{d\xi^2}\right)+\frac{d}{d\xi}\left(B_{yz}\frac{d^2 g_i}{d\xi^2}\right)\right]=\delta_{i8} \tag{25}$$

where $\delta_{ij}$ is a Kronecker symbol. The shifting functions are derived and their physical meanings are discussed in the Appendix. After substituting Eqs. (15–25) into Eqs. (3–13), one has the following differential equations in terms of $\bar{w}(\xi,\tau)$ and $\bar{v}(\xi,\tau)$:

$$\frac{\partial^2}{\partial\xi^2}\left(B_{yy}\frac{\partial^2\bar{w}}{\partial\xi^2}\right)+\frac{\partial^2}{\partial\xi^2}\left(B_{yz}\frac{\partial^2\bar{v}}{\partial\xi^2}\right)+M\frac{\partial^2\bar{w}}{\partial\tau^2}=\bar{p}(\xi,\tau) \tag{26}$$

$$\frac{\partial^2}{\partial\xi^2}\left(B_{zz}\frac{\partial^2\bar{v}}{\partial\xi^2}\right)+\frac{\partial^2}{\partial\xi^2}\left(B_{yz}\frac{\partial^2\bar{w}}{\partial\xi^2}\right)+M\frac{\partial^2\bar{v}}{\partial\tau^2}=\bar{q}(\xi,\tau) \tag{27}$$

where

$$\bar{p}(\xi,\tau)=P(\xi,\tau)-M(\xi)\sum_{i=1}^{8}\frac{d^2\bar{F}_i}{d\tau^2}g_i(\xi)$$
$$\bar{q}(\xi,\tau)=Q(\xi,\tau)-M(\xi)\sum_{i=1}^{8}\frac{d^2\bar{F}_i}{d\tau^2}\bar{g}_i(\xi) \tag{28}$$

and the following associated homogeneous boundary conditions.

At $\xi=0$,

$$\gamma_{11}\frac{\partial\bar{w}}{\partial\xi}-\gamma_{12}\left(B_{yy}\frac{\partial^2\bar{w}}{\partial\xi^2}+B_{yz}\frac{\partial^2\bar{v}}{\partial\xi^2}\right)=0 \tag{29}$$

$$\gamma_{21}\bar{w}+\gamma_{22}\left[\frac{\partial}{\partial\xi}\left(B_{yy}\frac{\partial^2\bar{w}}{\partial\xi^2}\right)+\frac{\partial}{\partial\xi}\left(B_{yz}\frac{\partial^2\bar{v}}{\partial\xi^2}\right)\right]=0 \tag{30}$$

$$\gamma_{31}\frac{\partial\bar{v}}{\partial\xi}-\gamma_{32}\left(B_{zz}\frac{\partial^2\bar{v}}{\partial\xi^2}+B_{yz}\frac{\partial^2\bar{w}}{\partial\xi^2}\right)=0 \tag{31}$$

$$\gamma_{41}\bar{v}+\gamma_{42}\left[\frac{\partial}{\partial\xi}\left(B_{zz}\frac{\partial^2\bar{v}}{\partial\xi^2}\right)+\frac{\partial}{\partial\xi}\left(B_{yz}\frac{\partial^2\bar{w}}{\partial\xi^2}\right)\right]=0 \tag{32}$$

At $\xi=1$,

$$\gamma_{51}\frac{\partial\bar{w}}{\partial\xi}+\gamma_{52}\left(B_{yy}\frac{\partial^2\bar{w}}{\partial\xi^2}+B_{yz}\frac{\partial^2\bar{v}}{\partial\xi^2}\right)=0 \tag{33}$$

$$\gamma_{61}\bar{w}-\gamma_{62}\left[\frac{\partial}{\partial\xi}\left(B_{yy}\frac{\partial^2\bar{w}}{\partial\xi^2}\right)+\frac{\partial}{\partial\xi}\left(B_{yz}\frac{\partial^2\bar{v}}{\partial\xi^2}\right)\right]=0 \tag{34}$$

$$\gamma_{71}\frac{\partial\bar{v}}{\partial\xi}+\gamma_{72}\left(B_{zz}\frac{\partial^2\bar{v}}{\partial\xi^2}+B_{yz}\frac{\partial^2\bar{w}}{\partial\xi^2}\right)=0 \tag{35}$$

$$\gamma_{81}\bar{v}-\gamma_{82}\left[\frac{\partial}{\partial\xi}\left(B_{yy}\frac{\partial^2\bar{v}}{\partial\xi^2}\right)+\frac{\partial}{\partial\xi}\left(B_{yz}\frac{\partial^2\bar{w}}{\partial\xi^2}\right)\right]=0 \tag{36}$$

The transformed initial conditions (15) become

$$\bar{w}(\xi,0)=W_0(\xi)-\sum_{i=1}^{8}\bar{F}_i(0)g_i(\xi)$$
$$\bar{v}(\xi,0)=V_0(\xi)-\sum_{i=1}^{8}\bar{F}_i(0)\bar{g}_i(\xi)$$
$$\frac{\partial\bar{w}(\xi,0)}{\partial\tau}=\dot{W}_0(\xi)-\sum_{i=1}^{8}\frac{d\bar{F}_i(0)}{d\tau}g_i(\xi)$$
$$\frac{\partial\bar{v}(\xi,0)}{\partial\tau}=\dot{V}_0(\xi)-\sum_{i=1}^{8}\frac{d\bar{F}_i(0)}{d\tau}\bar{g}_i(\xi) \tag{37}$$

**Orthogonality Condition**

The solution for Eqs. (26–37), $\bar{w}(\xi,\tau)$, and $\bar{v}(\xi,\tau)$ can be obtained by using the method of eigenfunction expansion. The eigenfunctions are specified by the associated homogeneous governing differential equations and homogeneous boundary conditions.

To derive the orthogonality condition of the eigenfunctions of the system, one lets $\Lambda_n^2$ be the $n$th eignevalue or the square of the $n$th dimensionless natural frequency and $[\hat{w}_n(\xi) \quad \hat{v}_n(\xi)]^T$ be the $n$th eigenfunction of the system, where the superscript $T$ is the transpose of a matrix. The governing characteristic differential equation can be expressed as

$$\left\{[\hat{L}] - \Lambda_n^2[\hat{M}]\right\}\begin{bmatrix}\hat{w}_n\\ \hat{v}_n\end{bmatrix} = 0 \tag{38}$$

where the differential operators $[\hat{L}]$ and $[\hat{M}]$ are

$$[\hat{L}] = \begin{bmatrix} \dfrac{d^2}{d\xi^2}\left(B_{yy}\dfrac{d^2}{d\xi^2}\right) & \dfrac{d^2}{d\xi^2}\left(B_{yz}\dfrac{d^2}{d\xi^2}\right) \\ \dfrac{d^2}{d\xi^2}\left(B_{yz}\dfrac{d^2}{d\xi^2}\right) & \dfrac{d^2}{d\xi^2}\left(B_{zz}\dfrac{d^2}{d\xi^2}\right) \end{bmatrix} \tag{39}$$

and

$$[\hat{M}] = \begin{bmatrix} M(\xi) & 0 \\ 0 & M(\xi)\end{bmatrix} \tag{40}$$

respectively. The eigenfunctions satisfy the boundary conditions (29–36). It can be seen that Eq. (38) and the associated boundary conditions (29–36) take the meaning of the free vibration of an elastically restrained nonuniform beam. The eigenfunctions and the eigenvalues can be obtained by using the modified transfer matrix method proposed by Lin.[16] It was shown that, as the element of the field transfer matrix can be integrated anaytically, the exact field transfer matrix of the system is, therefore, obtained.

Taking the inner product, one can easily show that

$$\int_0^1 [\hat{w}_j(\xi) \quad \hat{v}_j(\xi)][\hat{M}]\begin{bmatrix}\hat{w}_n(\xi)\\ \hat{v}_n(\xi)\end{bmatrix} d\xi = \int_0^1 [\hat{w}_n(\xi) \quad \hat{v}_n(\xi)][\hat{M}]\begin{bmatrix}\hat{w}_j(\xi)\\ \hat{v}_j(\xi)\end{bmatrix} d\xi \tag{41}$$

and

$$\int_0^1 [\hat{w}_j(\xi) \quad \hat{v}_j(\xi)][\hat{L}]\begin{bmatrix}\hat{w}_n(\xi)\\ \hat{v}_n(\xi)\end{bmatrix} d\xi = \int_0^1 [\hat{w}_n(\xi) \quad \hat{v}_n(\xi)][\hat{L}]\begin{bmatrix}\hat{w}_j(\xi)\\ \hat{v}_j(\xi)\end{bmatrix} d\xi + \hat{B} \tag{42}$$

where

$$\begin{aligned}\hat{B} &= \hat{w}_n\left[\frac{d}{d\xi}\left(B_{yy}\frac{d^2\hat{w}_j}{d\xi^2}\right) + \frac{d}{d\xi}\left(B_{yz}\frac{d^2\hat{v}_j}{d\xi^2}\right)\right]\Bigg|_0^1 \\ &+ \hat{v}_n\left[\frac{d}{d\xi}\left(B_{yz}\frac{d^2\hat{w}_j}{d\xi^2}\right) + \frac{d}{d\xi}\left(B_{zz}\frac{d^2\hat{v}_j}{d\xi^2}\right)\right]\Bigg|_0^1 \\ &- \frac{d\hat{w}_n}{d\xi}\left[B_{yy}\frac{d^2\hat{w}_j}{d\xi^2} + B_{yz}\frac{d^2\hat{v}_j}{d\xi^2}\right]\Bigg|_0^1 - \frac{d\hat{v}_n}{d\xi}\left[B_{yz}\frac{d^2\hat{w}_j}{d\xi^2} + B_{zz}\frac{d^2\hat{v}_j}{d\xi^2}\right]\Bigg|_0^1 \\ &+ \frac{d\hat{w}_j}{d\xi}\left[B_{yy}\frac{d^2\hat{w}_n}{d\xi^2} + B_{yz}\frac{d^2\hat{v}_n}{d\xi^2}\right]\Bigg|_0^1 + \frac{d\hat{v}_j}{d\xi}\left[B_{yz}\frac{d^2\hat{w}_n}{d\xi^2} + B_{zz}\frac{d^2\hat{v}_n}{d\xi^2}\right]\Bigg|_0^1 \\ &- \hat{w}_j\left[\frac{d}{d\xi}\left(B_{yy}\frac{d^2\hat{w}_n}{d\xi^2}\right) + \frac{d}{d\xi}\left(B_{yz}\frac{d^2\hat{v}_n}{d\xi^2}\right)\right]\Bigg|_0^1 \\ &- \hat{v}_j\left[\frac{d}{d\xi}\left(B_{yz}\frac{d^2\hat{w}_n}{d\xi^2}\right) + \frac{d}{d\xi}\left(B_{zz}\frac{d^2\hat{v}_n}{d\xi^2}\right)\right]\Bigg|_0^1 \end{aligned} \tag{43}$$

and $\hat{B}$ vanishes because of the boundary conditions (29–36). Thus, the self-adjointness of the system is proved. Consequently, the following orthogonality condition is obtained:

$$\int_0^1 M(\xi)[\hat{w}_j(\xi)\hat{w}_n(\xi) + \hat{v}_j(\xi)\hat{v}_n(\xi)]\, d\xi = \begin{cases} 0, & j \neq n \\ \varepsilon_n, & j = n\end{cases} \tag{44}$$

where $\varepsilon_n$ is a real number.

**Mode Superposition**

The solution $\bar{w}(\xi,\tau)$ and $\bar{v}(\xi,\tau)$ specified by Eqs. (26–37) can be expressed in the following eigenfunction expansion form:

$$\begin{bmatrix}\bar{w}(\xi,\tau)\\ \bar{v}(\xi,\tau)\end{bmatrix} = \sum_{n=1}^{\infty} T_n(\tau)\begin{bmatrix}\hat{w}_n(\xi)\\ \hat{v}_n(\xi)\end{bmatrix} \tag{45}$$

Substituting it back to the governing equations (26–28) and the initial conditions (37), multiplying by $[\hat{w}_n(\xi) \quad \hat{v}_n(\xi)]$, and integrating in accordance with the orthogonality condition (44), one obtains

$$\frac{d^2 T_n}{d\tau^2} + \Lambda_n^2 T_n = \frac{1}{\varepsilon_n}\int_0^1 [\hat{w}_n(\xi)\bar{p}(\xi,\tau) + \hat{v}_n(\xi)\bar{q}(\xi,\tau)]\, d\xi \tag{46}$$

The corresponding initial conditions are

$$T_n(0) = \frac{1}{\varepsilon_n}\int_0^1 M(\xi)[\hat{w}_n(\xi)\bar{w}(\xi,0) + \hat{v}_n(\xi)\bar{v}(\xi,0)]\, d\xi \tag{47}$$

$$\frac{dT_n(0)}{d\tau} = \frac{1}{\varepsilon_n}\int_0^1 M(\xi)\left[\hat{w}_n(\xi)\frac{\partial\bar{w}(\xi,0)}{\partial\tau} + \hat{v}_n(\xi)\frac{\partial\bar{v}(\xi,0)}{\partial\tau}\right] d\xi \tag{48}$$

The solution is

$$\begin{aligned} T_n(\tau) &= T_n(0)\cos\Lambda_n\tau + \frac{1}{\Lambda_n}\frac{dT_n(0)}{d\tau}\sin\Lambda_n\tau \\ &+ \frac{1}{\Lambda_n}\int_0^\tau p_n^*(\zeta)\sin\Lambda_n(\tau-\zeta)\, d\zeta \end{aligned} \tag{49}$$

where $p_n^*(\xi)$ is the forced term in Eq. (46). After substituting Eq. (49) back into Eq. (45), the general forced response of the beam with time-dependent boundary conditions is finally obtained by substituting the shifting functions (A6) and (A8) listed in the Appendix and the solution (45) into Eqs. (15).

## Results and Discussion

To illustrate the application of the method and explore the physical phenomena of the system, the following examples are presented.

**Example 1**

Consider the vibration of a clamped–hinged uniform untwisted beam subjected to a displacement time-dependent excitation $F_6 = 0.01\tau^2$ at the right end of the beam. For convenience, one takes the initial conditions as $W_0(\xi) = \dot{W}_0(\xi) = 0$ and transverse forces $P(\xi,\tau) = Q(\xi,\tau) = 0$. Therefore, $F_i = 0, i \neq 6$, and $F_j^* = 0, j = 1, 2, \ldots, 7, 8$. The governing characteristic differential equation (38) is reduced to

$$\frac{d^4\hat{w}_n}{d\xi^4} - \Lambda_n^2\hat{w}_n = 0 \tag{50}$$

Its four fundamental solutions are obtained:

$$\begin{aligned} \hat{w}_{1,n} &= \tfrac{1}{2}\left(\cosh\sqrt{\Lambda_n}\xi + \cos\sqrt{\Lambda_n}\xi\right) \\ \hat{w}_{2,n} &= \left(1/2\sqrt{\Lambda_n}\right)\left(\sinh\sqrt{\Lambda_n}\xi + \sin\sqrt{\Lambda_n}\xi\right) \\ \hat{w}_{3,n} &= \left(1/2\Lambda_n\right)\left(\cosh\sqrt{\Lambda_n}\xi - \cos\sqrt{\Lambda_n}\xi\right) \\ \hat{w}_{4,n} &= \left(1/2\Lambda_n^{3/2}\right)\left(\sinh\sqrt{\Lambda_n}\xi - \sin\sqrt{\Lambda_n}\xi\right) \end{aligned} \tag{51}$$

which satisfy the normalization condition

$$\begin{bmatrix} \hat{w}_{1,n}(0) & \hat{w}_{2,n}(0) & \hat{w}_{3,n}(0) & \hat{w}_{4,n}(0) \\ \hat{w}'_{1,n}(0) & \hat{w}'_{2,n}(0) & \hat{w}'_{3,n}(0) & \hat{w}'_{4,n}(0) \\ \hat{w}''_{1,n}(0) & \hat{w}''_{2,n}(0) & \hat{w}''_{3,n}(0) & \hat{w}''_{4,n}(0) \\ \hat{w}'''_{1,n}(0) & \hat{w}'''_{2,n}(0) & \hat{w}'''_{3,n}(0) & \hat{w}'''_{4,n}(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{52}$$

The homogeneous solution of Eq. (50) can be expressed as

$$\hat{w}_n(\xi) = c_{1,n}\hat{w}_{1,n} + c_{2,n}\hat{w}_{2,n} + c_{3,n}\hat{w}_{3,n} + c_{4,n}\hat{w}_{4,n} \tag{53}$$

where $c_{1,n}$, $c_{2,n}$, $c_{3,n}$, and $c_{4,n}$ are four constants to be determined. After substituting Eq. (53) into the associated boundary conditions, the following frequency equation is obtained:

$$\hat{w}_{3,n}(1)\hat{w}''_{4,n}(1) - \hat{w}_{4,n}(1)\hat{w}''_{3,n}(1) = 0 \tag{54}$$

The eigenvalues of the beam can be obtained by finding the roots of the frequency equation. The transient solution can be expressed by using the proposed method as follows:

$$W(\xi,\tau) = 0.01\tau^2 g_6(\xi) + \sum_{i=1}^{\infty} \frac{P_i^*}{\Lambda_i^2}(1 - \cos\Lambda_i\tau)\hat{w}_i(\xi) \tag{55}$$

where

$$g_6 = \frac{3}{2}\xi^2 - \frac{1}{2}\xi^3$$
$$P_i^* = -0.02\left[\int_0^1 \hat{w}_n(\xi)g_6(\xi)\,\mathrm{d}\xi\right] \bigg/ \left[\int_0^1 \hat{w}_n^2(\xi)\,\mathrm{d}\xi\right] \tag{56}$$

Its numerical results are listed in rows a of Table 1. The results in rows b are determined from Eq. (55), with the eigenvalues $\Lambda_i$ and the eigenfunctions $\hat{w}_i$ derived by using the modified transfer matrix method given by Lin.[16] Those in rows c and d are obtained by Lee and Lin[5] and Grant,[4] respectively. As can be seen, these results are very consistent.

**Example 2**

To establish the element stiffness matrix of a nonuniform beam with arbitrary pretwist, the static deflection curves of the beam subjected only to a unit displacement or a unit slope at either end of the beam segment have to be determined. However, it is known from the meanings of the shifting functions explored in the Appendix that these deflection curves are just the shifting functions $g_i(\xi)$ and $\bar{g}_i(\xi)$ for the clamped-clamped beam listed in case 3 of the Appendix. Thus, the element stiffness matrix relation can be written as

$$[-Q_z(0) M_z(0) Q_z(1) - M_z(1) - Q_y(0) M_y(0) Q_y(1) - M_y(1)]^T$$
$$= [k_{ij}]_{8\times 8}\left[\frac{\mathrm{d}w^*(0)}{\mathrm{d}\xi} w^*(0) \frac{\mathrm{d}v^*(0)}{\mathrm{d}\xi} v^*(0) \frac{\mathrm{d}w^*(1)}{\mathrm{d}\xi} w^*(1) \frac{\mathrm{d}v^*(1)}{\mathrm{d}\xi} v^*(1)\right]^T \tag{57}$$

**Table 1 Dynamic response of clamped–hinged uniform untwisted beams subjected to a displacement excitation at the right end[a]**

| $\tau$ | | $\xi = 0.2$ | $\xi = 0.4$ | $\xi = 0.6$ | $\xi = 0.8$ | $\xi = 1.0$ |
|---|---|---|---|---|---|---|
| 1 | a | 0.0006 | 0.0021 | 0.0043 | 0.0070 | 0.0100 |
| | b | 0.0005 | 0.0020 | 0.0042 | 0.0070 | 0.0100 |
| | c | 0.0006 | 0.0020 | 0.0043 | 0.0070 | 0.0100 |
| | d | 0.0005 | 0.0020 | 0.0042 | 0.0070 | 0.0100 |
| 3 | a | 0.0050 | 0.0187 | 0.0389 | 0.0634 | 0.0900 |
| | b | 0.0050 | 0.0187 | 0.0388 | 0.0633 | 0.0900 |
| | c | 0.0050 | 0.0187 | 0.0389 | 0.0634 | 0.0900 |
| | d | 0.0050 | 0.0187 | 0.0389 | 0.0633 | 0.0900 |
| 5 | a | 0.0140 | 0.0520 | 0.1080 | 0.1760 | 0.2500 |
| | b | 0.0140 | 0.0520 | 0.1080 | 0.1760 | 0.2500 |
| | c | 0.0140 | 0.0520 | 0.1080 | 0.1760 | 0.2500 |
| | d | 0.0140 | 0.0520 | 0.1080 | 0.1760 | 0.2500 |

[a] $[F_6 = 0.01\tau^2, F_i = 0, i \neq 6, F_j^* = 0, j = 1, 2, \ldots, 7, 8$, and $W_0(\xi) = \dot{W}_0(\xi) = P(\xi,\tau) = Q(\xi,\tau) = 0]$

where $v^*$ and $w^*$ represent the static displacements in the $y$ and $z$ directions, respectively, and the elements of the stiffness matrix are

$$k_{1i} = c_{1,i}, \qquad k_{2i} = -c_{2,i}, \qquad k_{3i} = -c_{1,i}, \qquad k_{4i} = c_{1,i} + c_{2,i}$$
$$k_{5i} = c_{3,i}, \qquad k_{6i} = -c_{4,i}, \qquad k_{7i} = -c_{3,i}, \qquad k_{8i} = c_{3,i} + c_{4,i} \tag{58}$$

in which the coefficients $c_{j,i}$ are the coefficients of the shifting functions. If the functions in Eq. (A9) can be integrated analytically, the exact stiffness matrix is obtained. Otherwise, an accurate solution can be easily obtained by using the numerical integration method. If the beam is unpretwisted, the following reduced matrix relation, which is the same as that given by Friedman and Kosmatka,[17] can be obtained:

$$\begin{bmatrix} -Q_z(0) \\ M_z(0) \\ Q_z(1) \\ -M_z(1) \end{bmatrix} = \frac{1}{\kappa}\begin{bmatrix} \omega_2(1) & \omega_1(1) & -\omega_2(1) & w_2(1) \\ \omega_1(1) & \omega_1(1) - w_1(1) & -\omega_1(1) & w_1(1) \\ -\omega_2(1) & -\omega_1(1) & \omega_2(1) & -w_2(1) \\ w_2(1) & w_1(1) & -w_2(1) & w_2(1) - w_1(1) \end{bmatrix} \times \begin{bmatrix} w^*(0) \\ \dfrac{\mathrm{d}w^*(0)}{\mathrm{d}\xi} \\ w^*(1) \\ \dfrac{\mathrm{d}w^*(1)}{\mathrm{d}\xi} \end{bmatrix} \tag{59}$$

where $\kappa = \omega_1(1)w_2(1) - \omega_2(1)w_1(1)$. Moreover, if the beam is of uniform cross section, the matrix relation (59) is further reduced to be the same as that given by Paz.[18]

**Example 3**

We investigate the vibration control of a pretwisted beam with boundary inputs. It is assumed that the beam is subjected to external harmonic transverse concentrated forces. The frequency of the boundary control inputs is the same as that of the external forces. The external forces and boundary inputs are

$$P(\xi,\tau) = \hat{P}\sin\varpi\tau\,\delta(\xi - \xi_0), \qquad Q(\xi,\tau) = \hat{Q}\sin\varpi\tau\,\delta(\xi - \xi_0)$$
$$\bar{F}_i(\tau) = \hat{F}_i\sin\varpi\tau, \qquad i = 1, 2, \ldots, 8 \tag{60}$$

where the coefficients $\hat{P}$ and $\hat{Q}$ are given and $\hat{F}_i$ are to be determined. If the transient response from the initial conditions is neglected, the general dynamic solution (15) is reduced to the following steady solution:

$$W(\xi,\tau) = \hat{W}(\xi)\sin\varpi\tau = \left[\hat{P}W_p^*(\xi) + \hat{Q}W_q^*(\xi) + \sum_{i=1}^{8}\hat{F}_i W_i^*(\xi)\right]\sin\varpi\tau$$
$$V(\xi,\tau) = \hat{V}(\xi)\sin\varpi\tau = \left[\hat{P}V_p^*(\xi) + \hat{Q}V_q^*(\xi) + \sum_{i=1}^{8}\hat{F}_i V_i^*(\xi)\right]\sin\varpi\tau \tag{61}$$

where

$$W_p^*(\xi) = \sum_{n=1}^{\infty}\frac{\bar{A}_{p,n}\hat{w}_n(\xi)}{\Lambda_n^2 - \varpi^2}, \qquad W_q^*(\xi) = \sum_{n=1}^{\infty}\frac{\bar{A}_{q,n}\hat{w}_n(\xi)}{\Lambda_n^2 - \varpi^2}$$
$$V_p^*(\xi) = \sum_{n=1}^{\infty}\frac{\bar{A}_{p,n}\hat{v}_n(\xi)}{\Lambda_n^2 - \varpi^2}, \qquad V_q^*(\xi) = \sum_{n=1}^{\infty}\frac{\bar{A}_{q,n}\hat{v}_n(\xi)}{\Lambda_n^2 - \varpi^2}$$
$$W_i^*(\xi) = g_i(\xi) + \sum_{n=1}^{\infty}\frac{\bar{A}_{i,n}\hat{w}_n(\xi)}{\Lambda_n^2 - \varpi^2}$$
$$V_i^*(\xi) = \bar{g}_i(\xi) + \sum_{n=1}^{\infty}\frac{\bar{A}_{i,n}\hat{v}_n(\xi)}{\Lambda_n^2 - \varpi^2} \tag{62}$$

in which

$$\bar{A}_{p,n} = \frac{\hat{w}_n(\xi_0)}{\varepsilon_n}, \qquad \bar{A}_{q,n} = \frac{\hat{v}_n(\xi_0)}{\varepsilon_n}$$

$$\bar{A}_{i,n} = \frac{\varpi^2}{\varepsilon_n}\int_0^1 M(\xi)[\hat{w}_n(\xi)g_i(\xi) + \hat{v}_n(\xi)\bar{g}_i(\xi)]\,\mathrm{d}\xi \qquad (63)$$

$$i = 1, 2, \ldots, 8$$

If the displacements at $\xi = \xi_1$ are controlled to be zero, the following two conditions are obtained from Eq. (61):

$$\sum_{i=1}^{8} \hat{F}_i W_i^*(\xi_1) = -\hat{P} W_p^*(\xi_1) - \hat{Q} W_q^*(\xi_1)$$

$$\sum_{i=1}^{8} \hat{F}_i V_i^*(\xi_1) = -\hat{P} V_p^*(\xi_1) - \hat{Q} V_q^*(\xi_1) \qquad (64)$$

One can choose only two boundary inputs to satisfy the conditions. If the $m$th and $n$th inputs are chosen, the corresponding coefficients of the boundary inputs are obtained:

$$\begin{bmatrix} \hat{F}_m \\ \hat{F}_n \end{bmatrix} = -\begin{bmatrix} W_m^*(\xi_1) & W_n^*(\xi_1) \\ V_m^*(\xi_1) & V_n^*(\xi_1) \end{bmatrix}^{-1} \begin{bmatrix} \hat{P} W_p^*(\xi_1) + \hat{Q} W_q^*(\xi_1) \\ \hat{P} V_p^*(\xi_1) + \hat{Q} V_q^*(\xi_1) \end{bmatrix} \qquad (65)$$

$$\hat{F}_i = 0, \qquad i \neq m, n$$

Similarly, if the displacements at $\xi = \xi_1$ and $\xi_2$ are controlled to be zero, one can choose the $j$th, $k$th, $m$th, and $n$th boundary inputs to satisfy the conditions. The corresponding coefficients of the boundary inputs are obtained:

$$\begin{bmatrix} \hat{F}_j \\ \hat{F}_k \\ \hat{F}_m \\ \hat{F}_n \end{bmatrix} = -\begin{bmatrix} W_j^*(\xi_1) & W_k^*(\xi_1) & W_m^*(\xi_1) & W_n^*(\xi_1) \\ W_j^*(\xi_2) & W_k^*(\xi_2) & W_m^*(\xi_2) & W_n^*(\xi_2) \\ V_j^*(\xi_1) & V_k^*(\xi_1) & V_m^*(\xi_1) & V_n^*(\xi_1) \\ V_j^*(\xi_2) & V_k^*(\xi_2) & V_m^*(\xi_2) & V_n^*(\xi_2) \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \hat{P} W_p^*(\xi_1) + \hat{Q} W_q^*(\xi_1) \\ \hat{P} W_p^*(\xi_2) + \hat{Q} W_q^*(\xi_2) \\ \hat{P} V_p^*(\xi_1) + \hat{Q} V_q^*(\xi_1) \\ \hat{P} V_p^*(\xi_2) + \hat{Q} V_q^*(\xi_2) \end{bmatrix}, \qquad \hat{F}_i = 0$$

$$i \neq j, k, m, n \qquad (66)$$

If the eight coefficients $\hat{F}_i$ of the boundary control inputs are taken in a similar manner, the displacements at arbitrary four positions can be controlled to be zero.

For investigating the vibration control of the beam further, the steady response of three kinds of cantilever tapered beams with different pretwist angles, i.e., 1) $\theta = \xi\Phi$, 2) $\theta = \xi^2\Phi$, and 3) $\theta = \xi(2-\xi)\Phi$, subjected to a concentrated harmonic transverse force without boundary control is discussed. The corresponding steady response will be

$$W(\xi, \tau) = \hat{P} W_p^*(\xi)\sin\varpi\tau, \qquad V(\xi, \tau) = \hat{P} V_p^*(\xi)\sin\varpi\tau \qquad (67)$$

It is shown in Fig. 2 that the influence of the tip pretwist angle $\Phi$ on the tip amplitudes of case 3 is greatest and on those of case 2 is weakest. It is concluded that a greater variation of the pretwist angle near the root of the beam provides a greater influence for a given tip pretwist angle.

The vibration control of the cantilever tapered beam with the pretwist angle $\theta = \xi\pi$, discussed earlier, is investigated with three kinds of boundary control methods. First, if only the tip displacements are controlled to be zero and the second and fourth boundary inputs are taken, the corresponding coefficients of the boundary inputs are obtained from Eq. (65), i.e., $\hat{F}_2 = -0.04757$, $\hat{F}_4 = -0.01920$, $\hat{F}_j = 0$, $j \neq 2, 4$. Second, if the first and third boundary

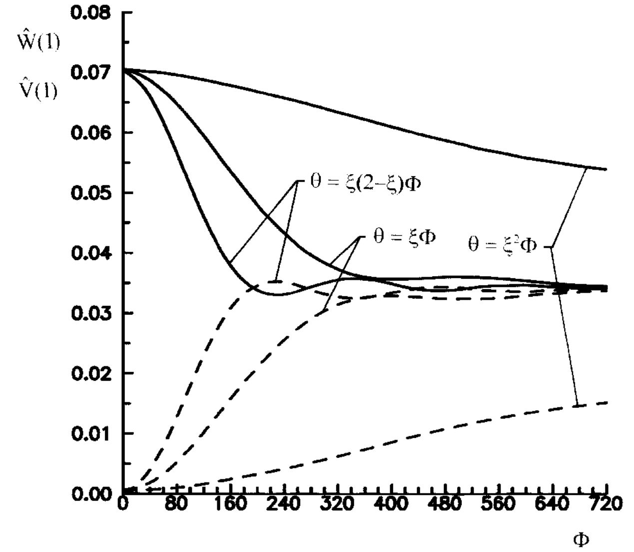


**Fig. 2 Influence of the tip pretwist angle $\Phi$ on the tip response of a cantilever beam subjected to a harmonic concentrated transverse force: $P(\xi, \tau) = 0.003\sin\tau\,\delta(\xi - 0.5)$, $Q(\xi, \tau) = 0$, $\bar{F}_i(\tau) = 0$, $B_{yy} = (1 - 0.1\xi)\cos^2\theta + 100(1 - 0.1\xi)^3\sin^2\theta$, $B_{zz} = 100(1 - 0.1\xi)^3\cos^2\theta + (1 - 0.1\xi)\sin^2\theta$, and $B_{yz} = [50(1 - 0.1\xi)^3 - 0.5(1 - 0.1\xi)]\sin^2\theta$: ——, $\hat{W}(1)$, and - - -, $\hat{V}(1)$.**

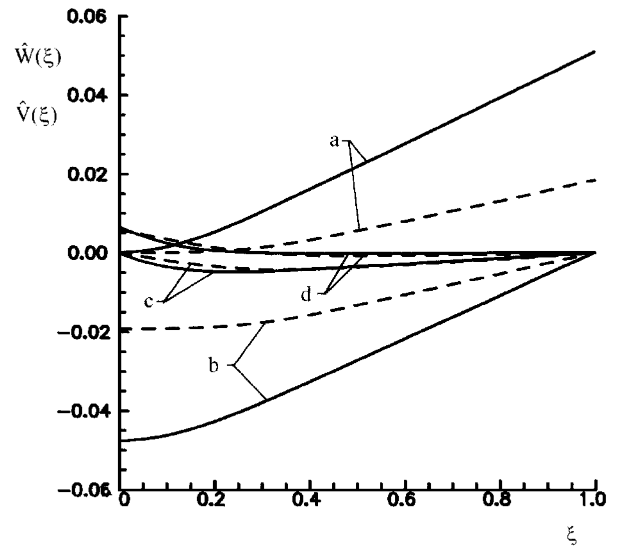


**Fig. 3 Influence of three kinds of boundary control methods on the displacements of a cantilever beam: $P(\xi, \tau) = 0.003\sin\tau\,\delta(\xi - 0.5)$, $Q(\xi, \tau) = 0$; ——, $\hat{W}(\xi)$; - - -, $\hat{V}(\xi)$; a, $\bar{F}_i(\tau) = 0$, $i = 1, 2, \ldots, 8$; b, $\hat{F}_2 = -0.04757$, $\hat{F}_4 = -0.01920$, $\hat{F}_j = 0$, $j \neq 2, 4$; c, $\hat{F}_1 = -0.04850$, $\hat{F}_3 = -0.01898$, $\hat{F}_j = 0$, $j \neq 1, 3$; and d, $\hat{F}_1 = -0.05526$, $\hat{F}_2 = 0.00665$, $\hat{F}_3 = -0.02491$, $\hat{F}_4 = 0.00594$, $\hat{F}_j = 0$, $j \neq 1, 2, 3, 4$.**

inputs are taken, the corresponding coefficients of the boundary inputs are also obtained from Eq. (65), i.e., $\hat{F}_1 = -0.04850$, $\hat{F}_3 = -0.01898$, $\hat{F}_j = 0$, $j \neq 1, 3$. Finally, if the displacements at $\xi = 0.3$ and 1 are controlled to be zero, the corresponding coefficients of the boundary inputs are obtained from Eq. (66), i.e., $\hat{F}_1 = -0.05526$, $\hat{F}_2 = 0.00665$, $\hat{F}_3 = -0.02491$, $\hat{F}_4 = 0.00594$, $\hat{F}_j = 0$, $j \neq 1, 2, 3, 4$. It is shown in Fig. 3 that the displacements obtained by using the first method are larger than those obtained by using the second one and the third one. Moreover, the displacements obtained by using the third method are smaller than those obtained by using the first one and the second one.

## Summary

The governing differential equations with the general time-dependent elastic boundary conditions for the coupled bending–bending vibration of a pretwisted nonuniform beam are derived by

Hamilton's principle. A solution procedure to study the dynamic response of the beam with arbitrary pretwist is proposed. The physical meanings of the shifting functions are revealed. The self-adjointness of the system is proved. The orthogonality condition for the eigenfunctions of a nonuniform pretwisted beam with elastic boundary conditions is derived. The stiffness matrix for a nonuniform beam with arbitrary pretwist is derived. As the coefficients of the matrix can be integrated analytically, the exact stiffness matrix is, therefore, obtained. The greater the variation is of the pretwist angle near the root of a cantilever beam subjected to a transverse force, the greater the influence of pretwist for a given tip pretwist angle. For the steady motion of a beam subjected to an external harmonic transverse force, its displacements at an arbitrary four or fewer positions can be controlled to be zero at the same time by taking boundary inputs.

## Appendix: Shifting Functions

### Case 1: General Case

The system composed of Eqs. (16–25) in terms of the shifting functions presents the static problem of a pretwisted nonuniform beam subjected to unit end restraints. The shifting functions $g_i$ and $\bar{g}_i$ are the static deflections in the $z$ and $y$ directions, respectively, of a generally elastically restrained pretwisted beam subjected to a unit transformed moment or a unit transformed shear force at the ends, respectively. When the rotational spring constant is infinity or zero, the unit transformed moment is a unit end slope or a unit end moment. When the translational spring constant is infinity or zero, the unit transformed shear force is a unit end displacement or a unit end shear force.

The corresponding negative shear forces and moments are obtained by integrating Eqs. (16) and (17) once and twice, respectively:

$$-Q_{z,i}(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{yy}\frac{\mathrm{d}^2 g_i}{\mathrm{d}\xi^2}\right) + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{yz}\frac{\mathrm{d}^2 \bar{g}_i}{\mathrm{d}\xi^2}\right) = c_{1,i} \tag{A1}$$

$$-M_{z,i}(\xi) = B_{yy}\frac{\mathrm{d}^2 g_i}{\mathrm{d}\xi^2} + B_{yz}\frac{\mathrm{d}^2 \bar{g}_i}{\mathrm{d}\xi^2} = c_{1,i}\xi + c_{2,i} \tag{A2}$$

$$-Q_{y,i}(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{zz}\frac{\mathrm{d}^2 \bar{g}_i}{\mathrm{d}\xi^2}\right) + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{yz}\frac{\mathrm{d}^2 g_i}{\mathrm{d}\xi^2}\right) = c_{3,i} \tag{A3}$$

$$-M_{y,i}(\xi) = B_{zz}\frac{\mathrm{d}^2 \bar{g}_i}{\mathrm{d}\xi^2} + B_{yz}\frac{\mathrm{d}^2 g_i}{\mathrm{d}\xi^2} = c_{3,i}\xi + c_{4,i} \tag{A4}$$

Obviously, the coefficients $c_1$, $c_2$, $c_3$, and $c_4$ are the negative shear forces and the negative moments in the $z$ and $y$ directions at the left end, respectively. The shifting functions can be obtained easily from Eqs. (A2) and (A4):

$$\frac{\mathrm{d}g_i}{\mathrm{d}\xi} = c_{1,i}\omega_1(\xi) + c_{2,i}\omega_2(\xi) + c_{3,i}\omega_3(\xi) + c_{4,i}\omega_4(\xi) + c_{5,i} \tag{A5}$$

$$g_i(\xi) = c_{1,i}w_1(\xi) + c_{2,i}w_2(\xi) + c_{3,i}w_3(\xi) + c_{4,i}w_4(\xi) + c_{5,i}\xi + c_{6,i} \tag{A6}$$

$$\frac{\mathrm{d}\bar{g}_i}{\mathrm{d}\xi} = c_{1,i}\upsilon_1(\xi) + c_{2,i}\upsilon_2(\xi) + c_{3,i}\upsilon_3(\xi) + c_{4,i}\upsilon_4(\xi) + c_{7,i} \tag{A7}$$

$$\bar{g}_i(\xi) = c_{1,i}v_1(\xi) + c_{2,i}v_2(\xi) + c_{3,i}v_3(\xi) + c_{4,i}v_4(\xi) + c_{7,i}\xi + c_{8,i} \tag{A8}$$

where

$$\omega_1(\xi) = \int_0^\xi \frac{\zeta B_{zz}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)}\,\mathrm{d}\zeta$$

$$\omega_2(\xi) = \int_0^\xi \frac{B_{zz}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)}\,\mathrm{d}\zeta$$

$$\omega_3(\xi) = \upsilon_1(\xi) = \int_0^\xi \frac{\zeta B_{yz}(\zeta)}{B_{yz}^2(\zeta) - B_{zz}(\zeta)B_{yy}(\zeta)}\,\mathrm{d}\zeta$$

$$\omega_4(\xi) = \upsilon_2(\xi) = \int_0^\xi \frac{B_{yz}(\zeta)}{B_{yz}^2(\zeta) - B_{zz}(\zeta)B_{yy}(\zeta)}\,\mathrm{d}\zeta$$

$$\upsilon_3(\xi) = \int_0^\xi \frac{\zeta B_{yy}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)}\,\mathrm{d}\zeta$$

$$\upsilon_4(\xi) = \int_0^\xi \frac{B_{yy}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)}\,\mathrm{d}\zeta$$

$$w_j(\xi) = \int_0^\xi \omega_j(\zeta)\,\mathrm{d}\zeta, \qquad v_j(\xi) = \int_0^\xi \upsilon_j(\zeta)\,\mathrm{d}\zeta$$

$$j = 1, 2, 3, 4 \tag{A9}$$

Substituting Eqs. (A1–A9) into Eqs. (18–25), the coefficients of the shifting functions are obtained:

$$\begin{bmatrix} c_{1,i} \\ c_{2,i} \\ c_{3,i} \\ c_{4,i} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}^{-1} \times \begin{bmatrix} \delta_{i5} - (\gamma_{51}/\gamma_{11})\delta_{i1} \\ \delta_{i6} - (\gamma_{61}/\gamma_{11})\delta_{i1} - (\gamma_{61}/\gamma_{21})\delta_{i2} \\ \delta_{i7} - (\gamma_{71}/\gamma_{31})\delta_{i3} \\ \delta_{i8} - (\gamma_{81}/\gamma_{31})\delta_{i3} - (\gamma_{81}/\gamma_{41})\delta_{i4} \end{bmatrix} \tag{A10}$$

$$c_{5,i} = (1/\gamma_{11})(\delta_{i1} + \gamma_{12}c_{2,i}), \qquad c_{6,i} = (1/\gamma_{21})(\delta_{i2} - \gamma_{22}c_{1,i})$$

$$c_{7,i} = (1/\gamma_{31})(\delta_{i3} + \gamma_{32}c_{4,i}), \qquad c_{8,i} = (1/\gamma_{41})(\delta_{i4} - \gamma_{42}c_{3,i})$$

where

$$a_{11} = \gamma_{51}\omega_1(1) + \gamma_{52}, \qquad a_{12} = \gamma_{51}\omega_2(1) + \gamma_{52} + (\gamma_{51}/\gamma_{11})\gamma_{12}$$

$$a_{13} = \gamma_{51}\omega_3(1), \qquad a_{14} = \gamma_{51}\omega_4(1)$$

$$a_{21} = \gamma_{61}w_1(1) - \gamma_{62} - (\gamma_{61}/\gamma_{21})\gamma_{22}$$

$$a_{22} = \gamma_{61}w_2(1) + (\gamma_{61}/\gamma_{11})\gamma_{12}, \qquad a_{23} = \gamma_{61}w_3(1)$$

$$a_{24} = \gamma_{61}w_4(1), \qquad a_{31} = \gamma_{71}\upsilon_1(1), \qquad a_{32} = \gamma_{71}\upsilon_2(1) \tag{A11}$$

$$a_{33} = \gamma_{71}\upsilon_3(1) + \gamma_{72}$$

$$a_{34} = \gamma_{71}\upsilon_4(1) + \gamma_{72} + (\gamma_{71}/\gamma_{31})\gamma_{32}, \qquad a_{41} = \gamma_{81}v_1(1)$$

$$a_{42} = \gamma_{81}v_2(1), \qquad a_{43} = \gamma_{81}v_3(1) - \gamma_{82} - (\gamma_{81}/\gamma_{41})\gamma_{42}$$

$$a_{44} = \gamma_{81}v_4(1) + (\gamma_{81}/\gamma_{31})\gamma_{32}$$

### Case 2: Clamped–Free

For this case, $\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = \gamma_{52} = \gamma_{62} = \gamma_{72} = \gamma_{82} = 1$, $\gamma_{12} = \gamma_{22} = \gamma_{32} = \gamma_{42} = \gamma_{51} = \gamma_{61} = \gamma_{71} = \gamma_{81} = 0$, and the shifting functions are

$$g_1 = \xi, \qquad g_2 = 1, \qquad g_3 = g_4 = 0, \qquad g_5 = w_2(\xi)$$
$$g_6 = -w_1(\xi), \qquad g_7 = w_4(\xi), \qquad g_8 = -w_3(\xi) + w_4(\xi) \tag{A12}$$

$$\bar{g}_1 = \bar{g}_2 = 0, \qquad \bar{g}_3 = \xi, \qquad \bar{g}_4 = 1, \qquad \bar{g}_5 = v_2(\xi)$$
$$\bar{g}_6 = v_1(\xi), \qquad \bar{g}_7 = v_4(\xi), \qquad \bar{g}_8 = -v_3(\xi) + v_4(\xi) \tag{A13}$$

### Case 3: Clamped–Clamped

For this case, $\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = \gamma_{51} = \gamma_{61} = \gamma_{71} = \gamma_{81} = 1$, $\gamma_{12} = \gamma_{22} = \gamma_{32} = \gamma_{42} = \gamma_{52} = \gamma_{62} = \gamma_{72} = \gamma_{82} = 0$, and the shifting functions are

$$g_1 = -(d_{11} + d_{12})w_1(\xi) - (d_{21} + d_{22})w_2(\xi) - (d_{31} + d_{32})w_3(\xi) - (d_{41} + d_{42})w_4(\xi) + \xi$$

$$g_2 = -[d_{12}w_1(\xi) + d_{22}w_2(\xi) + d_{32}w_3(\xi) + d_{42}w_4(\xi)] + 1$$

$$g_3 = -(d_{13}+d_{14})w_1(\xi) - (d_{23}+d_{24})w_2(\xi)$$
$$- (d_{33}+d_{34})w_3(\xi) - (d_{43}+d_{44})w_4(\xi)$$
$$g_4 = -[d_{14}w_1(\xi) + d_{24}w_2(\xi) + d_{34}w_3(\xi) + d_{44}w_4(\xi)]$$
$$g_{j+4} = d_{1j}w_1(\xi) + d_{2j}w_2(\xi) + d_{3j}w_3(\xi) + d_{4j}w_4(\xi)$$
$$j = 1, 2, 3, 4 \quad \text{(A14)}$$

$$\bar{g}_1 = -(d_{11}+d_{12})v_1(\xi) - (d_{21}+d_{22})v_2(\xi)$$
$$-(d_{31}+d_{32})v_3(\xi) - (d_{41}+d_{42})v_4(\xi)$$
$$\bar{g}_2 = -[d_{12}v_1(\xi) + d_{22}v_2(\xi) + d_{32}v_3(\xi) + d_{42}v_4(\xi)]$$
$$\bar{g}_3 = -(d_{13}+d_{14})v_1(\xi) - (d_{23}+d_{24})v_2(\xi)$$
$$- (d_{33}+d_{34})v_3(\xi) - (d_{43}+d_{44})v_4(\xi) + \xi \quad \text{(A15)}$$
$$\bar{g}_4 = -[d_{14}v_1(\xi) + d_{24}v_2(\xi) + d_{34}v_3(\xi) + d_{44}v_4(\xi)] + 1$$
$$\bar{g}_{j+4} = d_{1j}v_1(\xi) + d_{2j}v_2(\xi) + d_{3j}v_3(\xi) + d_{4j}v_4(\xi)$$
$$j = 1, 2, 3, 4$$

where

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix} = \begin{bmatrix} \omega_1(1) & \omega_2(1) & \omega_3(1) & \omega_4(1) \\ w_1(1) & w_2(1) & w_3(1) & w_4(1) \\ \upsilon_1(1) & \upsilon_2(1) & \upsilon_3(1) & \upsilon_4(1) \\ v_1(1) & v_2(1) & v_3(1) & v_4(1) \end{bmatrix}^{-1} \quad \text{(A16)}$$

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